

NON-EXPANSIVE BIJECTIONS BETWEEN UNIT BALLS OF BANACH SPACES

OLEZIA ZAVARZINA

ABSTRACT. It is known that if M is a finite-dimensional Banach space, or a strictly convex space, or the space ℓ_1 , then every non-expansive bijection $F : B_M \rightarrow B_M$ is an isometry. We extend these results to non-expansive bijections $F : B_E \rightarrow B_M$ between unit balls of two different Banach spaces. Namely, if E is an arbitrary Banach space and M is finite-dimensional or strictly convex, or the space ℓ_1 then every non-expansive bijection $F : B_E \rightarrow B_M$ is an isometry.

1. INTRODUCTION

Let M be a metric space. A map $F : M \rightarrow M$ is called *non-expansive*, if $\rho(F(x), F(y)) \leq \rho(x, y)$ for all $x, y \in M$. The space M is called *expand-contract plastic* (or simply, an EC-space) if every non-expansive bijection from M onto itself is an isometry.

It is known [8, Theorem 1.1] that every compact (or even totally bounded) metric space is expand-contract plastic, so in particular every bounded subset of \mathbb{R}^n is an EC-space.

The situation with bounded subsets of infinite dimensional spaces is different. On the one hand, there is a non-expand-contract plastic bounded closed convex subset of a Hilbert space [2, Example 2.7] (in fact, that set is an ellipsoid), but on the other hand, the unit ball of a Hilbert space, and in general the unit ball of every strictly convex Banach space is an EC-space [2, Theorem 2.6]. It is unknown whether the strict convexity condition in [2, Theorem 2.6] can be omitted, that is, in other words, the following problem arises.

Problem 1.1. *For what Banach spaces Y every bijective non-expansive map $F : B_Y \rightarrow B_Y$ is an isometry? Is this true for every Banach space?*

Outside of strictly convex spaces, Problem 1.1 is solved positively for all finite-dimensional spaces (because of the compactness of the unit ball), and for the space ℓ_1 [6, Theorem 1].

To the best of our knowledge, the following natural extension of Problem 1.1 is also open.

Problem 1.2. *Is it true, that for every pair (X, Y) of Banach spaces every bijective non-expansive map $F : B_X \rightarrow B_Y$ is an isometry?*

An evident bridge between these two problems is the third one, which we also are not able to solve.

2010 *Mathematics Subject Classification.* 46B20.

Key words and phrases. non-expansive map, unit ball, plastic space, strictly convex space.

Problem 1.3. *Let X, Y be Banach spaces that admit a bijective non-expansive map $F : B_X \rightarrow B_Y$. Is it true that spaces X and Y are isometric?*

In fact, if one solves Problem 1.2 in positive, one evidently solves also Problem 1.3. On the other hand, for a fixed pair (X, Y) the positive answers to Problems 1.1 and 1.3 would imply the same for Problem 1.2.

The aim of this article is to demonstrate that for all spaces Y where Problem 1.1 is known to have the positive solution (i.e. for strictly convex spaces, for ℓ_1 , and for finite-dimensional spaces), Problem 1.2 can be solved in positive for all pairs of the form (X, Y) . In reality, our result for pairs (X, Y) with Y being strictly convex repeats the corresponding proof of the case $X = Y$ from [2, Theorem 2.6] almost word-to-word. The proof for pairs (X, ℓ_1) on some stage needs additional work comparing to its particular case $X = \ell_1$ from [6, Theorem 1]. The most difficult one is the finite-dimensional case, where the approach from [8, Theorem 1.1] is not applicable for maps between two different spaces, because it uses iterations of the map. So, for finite-dimensional spaces we had to search for a completely different proof. Our proof in this case uses some ideas from [2] and [6] but elaborates them a lot.

There is another similar circle of problems that motivates our study.

In 1987, D. Tingley [11] proposed the following question: let f be a bijective isometry between the unit spheres S_X and S_E of real Banach spaces X, E respectively. Is it true that f extends to a linear (bijective) isometry $F : X \rightarrow E$ of the corresponding spaces?

Let us mention that this is equivalent to the fact that the following natural positive-homogeneous extension $F : X \rightarrow E$ of f is linear:

$$F(0) = 0, \quad F(x) = \|x\| f(x/\|x\|) \quad (x \in X \setminus \{0\}).$$

Since according to P. Mankiewicz's theorem [7] every bijective isometry between convex bodies can be uniquely extended to an affine isometry of the whole spaces, Tingley's problem can be reformulated as follows:

Problem 1.4. *Let $F : B_X \rightarrow B_E$ be a positive-homogeneous map, whose restriction to S_X is a bijective isometry between S_X and S_E . Is it true that F is an isometry itself?*

There is a number of publications devoted to Tingley's problem (see [3] for a survey of corresponding results) and, in particular, the problem is solved in positive for many concrete classical Banach spaces. Surprisingly, for general spaces this innocently-looking question remains open even in dimension two. For finite-dimensional polyhedral spaces the problem is solved in positive by V. Kadets and M. Martín in 2012 [5], and the positive solution for the class of generalized lush spaces was given by Dongni Tan, Xujian Huang, and Rui Liu in 2013 [10]. A step in the proof of the latter result was a lemma (Proposition 3.4 of [10]) which in our terminology says that if the map F in Problem 1.4 is non-expansive, then the problem has a positive solution. So, the problem which we address in our paper (Problem 1.2) can be considered as a much stronger variant of that lemma.

2. PRELIMINARIES

In the sequel, the letters X and Y always stand for real Banach spaces. We denote by S_X and B_X the unit sphere and the closed unit ball of X respectively. For a convex set $A \subset X$ denote by $\text{ext}(A)$ the set of extreme points of A ; that is, $x \in \text{ext}(A)$ if $x \in A$ and for every $y \in X \setminus \{0\}$ either $x + y \notin A$ or $x - y \notin A$. Recall that X is called strictly convex if all elements of S_X are extreme points of B_X , or in other words, S_X does not contain non-trivial line segments. Strict convexity of X is equivalent to the strict triangle inequality $\|x + y\| < \|x\| + \|y\|$ holding for all pairs of vectors $x, y \in X$ that do not have the same direction. For subsets $A, B \subset X$ we use the standard notation $A + B = \{x + y : x \in A, y \in B\}$ and $aA = \{ax : x \in A\}$. Now let us reformulate the results of [2] on the case of two different spaces. The following theorem generalizes [2, Theorem 2.3], where the case $X = Y$ was considered. It can be demonstrated repeating the proof of [2, Theorem 2.3] almost word to word.

Theorem 2.1. *Let $F : B_X \rightarrow B_Y$ be a non-expansive bijection. In the above notations the following hold.*

- (1) $F(0) = 0$.
- (2) $F^{-1}(S_Y) \subset S_X$.
- (3) If $F(x)$ is an extreme point of B_Y , then $F(ax) = aF(x)$ for all $a \in (0, 1)$.
- (4) If $F(x)$ is an extreme point of B_Y , then x is also an extreme point of B_X .
- (5) If $F(x)$ is an extreme point of B_Y , then $F(-x) = -F(x)$.

Moreover, if Y is strictly convex, then

- (i) F maps S_X bijectively onto S_Y ;
- (ii) $F(ax) = aF(x)$ for all $x \in S_X$ and $a \in (0, 1)$;
- (iii) $F(-x) = -F(x)$ for all $x \in S_X$.

Following notations from [2] for every $u \in S_X$ and $v \in X$ denote $u^*(v)$ the directional derivative of the function $x \mapsto \|x\|_X$ at the point u in the direction v :

$$u^*(v) = \lim_{a \rightarrow 0^+} \frac{1}{a} (\|u + av\|_X - \|u\|_X).$$

By the convexity of the function $x \mapsto \|x\|_X$, the directional derivative exists. If $E \subset X$ is a subspace and u is a smooth point of S_E then $u^*|_E$ (the restriction of u^* to E) is the unique norm-one linear functional on E that satisfies $u^*|_E(u) = 1$ (the supporting functional at point u). In general $u^* : X \rightarrow \mathbb{R}$ is not linear, but it is sub-additive, positively homogeneous and possesses the following property: for arbitrary $y_1, y_2 \in X$

$$u^*(y_1) - u^*(y_2) \leq \|y_1 - y_2\|_X. \quad (1)$$

The next lemma generalizes in a straightforward way [2, Lemma 2.4].

Lemma 2.2. *Let $F : B_X \rightarrow B_Y$ be a bijective non-expansive map, and suppose that for some $u \in S_X$ and $v \in B_X$ we have $u^*(-v) = -u^*(v)$, $\|F(u)\| = \|u\|$ and $F(av) = aF(v)$ for all $a \in [-1, 1]$. Then $(F(u))^*(F(v)) = u^*(v)$.*

The following result and Corollary 2.4 are extracted from the proof of [2, Lemma 2.5].

Lemma 2.3. *Let $F : B_X \rightarrow B_Y$ be a bijective non-expansive map such that $F(S_X) = S_Y$. Let $V \subset S_X$ be such a subset that $F(av) = aF(v)$ for all $a \in [-1, 1]$, $v \in V$. Denote $A = \{tx : x \in V, t \in [-1, 1]\}$, then $F|_A$ is a bijective isometry between A and $F(A)$.*

Proof. Fix arbitrary $y_1, y_2 \in A$. Let $E = \text{span}\{y_1, y_2\}$, and let $W \subset S_E$ be the set of smooth points of S_E (which is dense in S_E). All the functionals x^* , where $x \in W$, are linear on E , so $x^*(-y_i) = -x^*(y_i)$, for $i = 1, 2$. Also, according to our assumption, $F(ay_i) = aF(y_i)$ for all $a \in [-1, 1]$. Now we can apply Lemma 2.2.

$$\begin{aligned} \|F(y_1) - F(y_2)\|_Y &\leq \|y_1 - y_2\|_X = \sup\{x^*(y_1 - y_2) : x \in W\} \\ &= \sup\{x^*(y_1) - x^*(y_2) : x \in W\} \\ &= \sup\{(F(x))^*(F(y_1)) - (F(x))^*(F(y_2)) : x \in W\} \\ &\leq \|F(y_1) - F(y_2)\|_Y, \end{aligned}$$

where on the last step we used the inequality (1). So $\|F(y_1) - F(y_2)\| = \|y_1 - y_2\|$. \square

Corollary 2.4. *If $F : B_X \rightarrow B_Y$ is a bijective function that satisfies (i), (ii), and (iii) of Theorem 2.1, then F is an isometry.*

Proof. We can apply Lemma 2.3 with $V = S_X$ and $A = B_X$. \square

3. MAIN RESULTS

Theorem 3.1. *Let $F : B_X \rightarrow B_Y$ be a bijective non-expansive map. If Y is strictly convex, then F is an isometry.*

Proof. If Y is strictly convex, then F satisfies (i), (ii), and (iii) of Theorem 2.1, so Corollary 2.4 is applicable. \square

Our next goal is to show that each non-expansive bijection from the unit ball of arbitrary Banach space to the unit ball of ℓ_1 is an isometry. In the proof we will use the following three known results.

Proposition 3.2 (P. Mankiewicz's [7]). *If $A \subset X$ and $B \subset Y$ are convex with non-empty interior, then every bijective isometry $F : A \rightarrow B$ can be extended to a bijective affine isometry $\tilde{F} : X \rightarrow Y$.*

Taking into account that in the case of A, B being the unit balls every isometry maps 0 to 0, this result implies that every bijective isometry $F : B_X \rightarrow B_Y$ is the restriction of a linear isometry from X onto Y .

Proposition 3.3 (Brower's invariance of domain principle [1]). *Let U be an open subset of \mathbb{R}^n and $f : U \rightarrow \mathbb{R}^n$ be an injective continuous map, then $f(U)$ is open in \mathbb{R}^n .*

Proposition 3.4 (Proposition 4 of [6]). *Let X be a finite-dimensional normed space and V be a subset of B_X with the following two properties: V is homeomorphic to B_X and $V \supset S_X$. Then $V = B_X$.*

Now we give the promised theorem.

Theorem 3.5. *Let X be a Banach space, $F : B_X \rightarrow B_{\ell_1}$ be a bijective non-expansive map. Then F is an isometry.*

Proof. Denote $e_n = (\delta_{i,n})_{i \in \mathbb{N}}$, $n = 1, 2, \dots$ the elements of the canonic basis of ℓ_1 (here, as usual, $\delta_{i,n} = 0$ for $n \neq i$ and $\delta_{n,n} = 1$). It is well-known and easy to check that $\text{ext}(B_{\ell_1}) = \{\pm e_n, n = 1, 2, \dots\}$.

Denote $g_n = F^{-1}e_n$. According to item (4) of Theorem 2.1 each of g_n is an extreme point of B_X .

One more notation: for every $N \in \mathbb{N}$ and $X_N = \text{span}\{g_k\}_{k \leq N}$ denote U_N and ∂U_N the unit ball and the unit sphere of X_N respectively and analogously for $Y_N = \text{span}\{e_k\}_{k \leq N}$ denote V_N and ∂V_N the unit ball and the unit sphere of Y_N respectively.

Claim. *For every $N \in \mathbb{N}$ and every collection $\{a_k\}_{k \leq N}$ of reals with $\|\sum_{n \leq N} a_n g_n\| \leq 1$*

$$F\left(\sum_{n \leq N} a_n g_n\right) = \sum_{n \leq N} a_n e_n.$$

Proof of the Claim. We will use the induction in N . If $N = 1$, the Claim follows from items (3) and (5) of Theorem 2.1. Now assume the validity of the Claim for $N - 1$, and let us prove it for N . At first, for every $x = \sum_{i=1}^N \alpha_i g_i$ we will show that

$$\|x\| = \sum_{i=1}^N |\alpha_i|. \quad (2)$$

Note, that due to the positive homogeneity of norm, it is sufficient to consider $x = \sum_{i=1}^N \alpha_i g_i$, with $\sum_{i=1}^N |\alpha_i| \leq 1$. In such a case $x \in U_N$. So

$$\left\| \sum_{i=1}^{N-1} \alpha_i g_i \right\| \leq \sum_{i=1}^{N-1} \|\alpha_i g_i\| = \sum_{i=1}^{N-1} |\alpha_i| \leq \sum_{i=1}^N |\alpha_i| \leq 1,$$

and $\sum_{i=1}^{N-1} \alpha_i g_i \in U_N$. On the one hand,

$$\|x\| = \left\| \sum_{i=1}^N \alpha_i g_i \right\| \leq \sum_{i=1}^N |\alpha_i|.$$

On the other hand, by the inductive hypothesis $F(\sum_{i=1}^{N-1} \alpha_i g_i) = \sum_{i=1}^{N-1} \alpha_i e_i$ and by items (3) and (5) of Theorem 2.1 $F(-\alpha_N g_N) = -\alpha_N e_N$. Consequently,

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^{N-1} \alpha_i g_i + \alpha_N g_N \right\| = \rho\left(\sum_{i=1}^{N-1} \alpha_i g_i, (-\alpha_N g_N)\right) \\ &\geq \rho\left(F\left(\sum_{i=1}^{N-1} \alpha_i g_i\right), F(-\alpha_N g_N)\right) = \left\| \sum_{i=1}^{N-1} \alpha_i e_i + \alpha_N e_N \right\| = \sum_{i=1}^N |\alpha_i|, \end{aligned}$$

and (2) is demonstrated. That means that

$$U_N = \left\{ \sum_{n \leq N} a_n g_n : \sum_{n \leq N} |a_n| \leq 1 \right\}, \partial U_N = \left\{ \sum_{n \leq N} a_n g_n : \sum_{n \leq N} |a_n| = 1 \right\}.$$

The remaining part of the proof of the Claim, and of the whole theorem repeats almost literally the corresponding part of the proof of [6, Theorem 1], so we present it here only for the reader's convenience. Let us show that

$$F(U_N) \subset V_N. \quad (3)$$

To this end, consider $x \in U_N$. If x is of the form αg_N the statement follows from Theorem 2.1. So we must consider $x = \sum_{i=1}^N \alpha_i g_i$, $\sum_{i=1}^N |\alpha_i| \leq 1$ with $\sum_{i=1}^{N-1} |\alpha_i| \neq 0$. Denote the expansion of $F(x)$ by $F(x) = \sum_{i=1}^{\infty} y_i e_i$. For the element

$$x_1 = \frac{\sum_{i=1}^{N-1} \alpha_i g_i}{\sum_{i=1}^{N-1} |\alpha_i|}$$

by the induction hypothesis

$$F(x_1) = \frac{\sum_{i=1}^{N-1} \alpha_i e_i}{\sum_{i=1}^{N-1} |\alpha_i|}.$$

So we may write the following chain of inequalities:

$$\begin{aligned} 2 &= \left\| F(x_1) - \frac{\alpha_N}{|\alpha_N|} e_N \right\| \leq \left\| F(x_1) - \sum_{i=1}^N y_i e_i \right\| + \left\| \sum_{i=1}^N y_i e_i - \frac{\alpha_N}{|\alpha_N|} e_N \right\| \\ &= \|F(x_1) - F(x)\| + \left\| F(x) - \frac{\alpha_N}{|\alpha_N|} e_N \right\| - 2 \sum_{i=N+1}^{\infty} |y_i| \\ &\leq \|F(x_1) - F(x)\| + \left\| F(x) - F\left(\frac{\alpha_N}{|\alpha_N|} g_N\right) \right\| \leq \|x_1 - x\| + \left\| x - \frac{\alpha_N}{|\alpha_N|} g_N \right\| \\ &= \sum_{j=1}^{N-1} \left| \alpha_j - \frac{\alpha_j}{\sum_{i=1}^{N-1} |\alpha_i|} \right| + |\alpha_N| + \sum_{j=1}^{N-1} |\alpha_j| + \left| \alpha_N - \frac{\alpha_N}{|\alpha_N|} \right| \\ &= \sum_{j=1}^{N-1} |\alpha_j| \left(1 + \left| 1 - \frac{1}{\sum_{i=1}^{N-1} |\alpha_i|} \right| \right) + |\alpha_N| \left(1 + \left| 1 - \frac{1}{|\alpha_N|} \right| \right) = 2. \end{aligned}$$

This means that all the inequalities in between are in fact equalities, so in particular $\sum_{i=N+1}^{\infty} |y_i| = 0$, i.e. $F(x) = \sum_{i=1}^N y_i e_i \in V_N$ and (3) is proved.

Now, let us demonstrate that

$$F(U_N) \supset \partial V_N. \quad (4)$$

Assume to the contrary, that there is a $y \in \partial V_N \setminus F(U_N)$. Denote $x = F^{-1}(y)$. Then, $\|x\| = 1$ (by (2) of Theorem 2.1) and $x \notin U_N$. For every $t \in [0, 1]$ consider

$F(tx)$. Let $F(tx) = \sum_{n \in \mathbb{N}} b_n e_n$ be the corresponding expansion. Then,

$$\begin{aligned} 1 &= \|0 - tx\| + \|tx - x\| \geq \|0 - F(tx)\| + \|F(tx) - y\| \\ &= 2 \sum_{n > N} |b_n| + \left\| \sum_{n \leq N} b_n e_n \right\| + \left\| y - \sum_{n \leq N} b_n e_n \right\| \geq 2 \sum_{n > N} |b_n| + 1, \end{aligned}$$

so $\sum_{n > N} |b_n| = 0$. This means that $F(tx) \in V_N$ for every $t \in [0, 1]$. On the other hand, $F(U_N)$ contains a relative neighborhood of 0 in V_N (here we use that $F(0) = 0$ and Proposition 3.3), so the continuous curve $\{F(tx) : t \in [0, 1]\}$ in V_N which connects 0 and y has a non-trivial intersection with $F(U_N)$. This implies that there is a $t \in (0, 1)$ such that $F(tx) \in F(U_N)$. Since $tx \notin U_N$ this contradicts the injectivity of F . Inclusion (4) is proved.

Now, inclusions (3) and (4) together with Proposition 3.4 imply $F(U_N) = V_N$. Remark, that by (2) U_N is isometric to V_N and, by finite dimensionality, U_N and V_N are compacts. So, U_N and V_N can be considered as two copies of one the same compact metric space, and Theorem 1.1 of [8] implies that every bijective non-expansive map from U_N onto V_N is an isometry. In particular, F maps U_N onto V_N isometrically. Finally, the application of Proposition 3.2 gives us that the restriction of F to U_N extends to a linear map from X_N to Y_N , *which completes the proof of the Claim.*

Now let us complete the proof of the theorem. At first, passing in (2) to limit as $N \rightarrow \infty$ we get

$$\|z\| = \sum_{i=1}^{\infty} |z_i|$$

for every $z = \sum_{n=1}^{\infty} z_n g_n$ with $\sum_{n=1}^{\infty} |z_n| < \infty$. The continuity of F and the claim imply that for every $x = \sum_{n=1}^{\infty} x_n e_n \in B_{\ell_1}$

$$F\left(\sum_{n=1}^{\infty} x_n g_n\right) = \sum_{n=1}^{\infty} x_n e_n, \quad \text{so} \quad F^{-1}\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n g_n.$$

Consequently, for every $x, y \in B_{\ell_1}$, $x = \sum_{n=1}^{\infty} x_n e_n$ and $y = \sum_{n=1}^{\infty} y_n e_n$ the following equalities hold true:

$$\begin{aligned} \|x - y\| &= \left\| \sum_{n=1}^{\infty} (x_n - y_n) e_n \right\| = \sum_{n=1}^{\infty} |(x_n - y_n)| = \left\| \sum_{n=1}^{\infty} (x_n - y_n) g_n \right\| \\ &= \left\| \sum_{n=1}^{\infty} x_n g_n - \sum_{n=1}^{\infty} y_n g_n \right\| = \|F^{-1}(x) - F^{-1}(y)\|. \end{aligned}$$

So, F^{-1} is an isometry, consequently the same is true for F . \square

Our next (and the last) goal is to demonstrate that each non-expansive bijection between two different finite dimensional Banach spaces is an isometry. Below we recall the definitions and well-known properties of total and norming subsets of dual spaces that we will need further.

A subset $V \subset S_{X^*}$ is called *total* if for every $x \neq 0$ there exists $f \in V$ such that $f(x) \neq 0$. V is called *norming* if $\sup_{f \in V} |f(x)| = \|x\|$ for all $x \in X$. We will use the following easy exercise.

Lemma 3.6 ([4], Exercise 9, p. 538). *Let $A \subset S_X$ be dense in S_X , and for every $a \in A$ let f_a be a supporting functional at a . Then $V = \{f_a : a \in A\}$ is norming (and consequently total).*

The following known fact is an easy consequence of the bipolar theorem.

Lemma 3.7. *Let X be a reflexive space. Then $V \subset S_{X^*}$ is norming if and only if $\overline{\text{aconv}}(V) = B_{X^*}$.*

Now we can demonstrate the promised result.

Theorem 3.8. *Let X, Y be Banach spaces, Y be finite-dimensional, $F : B_X \rightarrow B_Y$ be a bijective non-expansive map. Then F is an isometry.*

Proof. Take an arbitrary finite-dimensional subspace $Z \subset X$. Then the restriction of F to B_Z is a bijective and continuous map between two compact sets B_Z and $F(B_Z)$, so B_Z and $F(B_Z)$ are homeomorphic. Thus, Brower's invariance of domain principle (Proposition 3.3) implies that $\dim Z \leq \dim Y$. By arbitrariness of $Z \subset X$ this implies that $\dim X \leq \dim Y$. Consequently, F being bijective and continuous map between compact sets B_X and B_Y , is a homeomorphism. Another application of Proposition 3.3 says that $\dim X = \dim Y$, F maps interior points in interior points, and $F(S_X) = S_Y$.

Let G be the set of all $x \in S_X$ such that norm is differentiable both at x and $F(x)$. According to [9, Theorem 25.5], the complement to the set of differentiability points of the norm is meager. Consequently, G being an intersection of two comeager sets, is dense in S_X . Recall that F is a homeomorphism, so $F(G)$ is dense in S_Y . Thus, Lemma 3.6 ensures that $A := \{x^* : x \in G\}$ and $B := \{F(x)^* : x \in G\} = \{y^* : y \in F(G)\}$ are norming subsets of X^* and Y^* respectively, and consequently by Lemma 3.7

$$\overline{\text{aconv}}(A) = B_{X^*}, \overline{\text{aconv}}(B) = B_{Y^*} \quad (5)$$

Denote $K = F^{-1}(\text{ext } B_Y) \subset \text{ext } B_X$. Note that for all $x \in G$ the corresponding $(F(x))^*$ and x^* are linear, and Lemma 2.2 implies that for all $x \in G$ and $z \in K$ the following equality holds true:

$$(F(x))^*(F(z)) = x^*(z).$$

Let us define the map $H : \{x^* : x \in G\} \rightarrow \{(F(x))^* : x \in G\}$ such that $H(x^*) = (F(x))^*$. For the correctness of this definition it is necessary to verify for all $x_1, x_2 \in G$ the implication

$$(x_1^* = x_2^*) \implies (F(x_1)^* = F(x_2)^*).$$

Assume for given $x_1, x_2 \in G$ that $x_1^* = x_2^*$. In order to check equality $F(x_1)^* = F(x_2)^*$ it is sufficient to verify that $F(x_1)^*y = F(x_2)^*y$ for $y \in \text{ext } B_Y$, i.e. for y of the form $y = F(x)$ with $x \in K$. Indeed,

$$F(x_1)^*(F(x)) = x_1^*(x) = x_2^*(x) = F(x_2)^*(F(x)).$$

Let us extend H by linearity to $\tilde{H} : X^* = \text{span}(x^*, x \in G) \rightarrow Y^*$. For $x^* = \sum_{k=1}^N \lambda_k x_k^*$, $x_k \in G$ let $\tilde{H}(x^*) = \sum_{k=1}^N \lambda_k H(x_k^*)$. To verify the correctness of this extension we will prove that

$$\left(\sum_{k=1}^N \lambda_k x_k^* = \sum_{k=1}^M \mu_k y_k^* \right) \implies \left(\sum_{k=1}^N \lambda_k H(x_k^*) = \sum_{k=1}^M \mu_k H(y_k^*) \right).$$

Again we will prove equality $\sum_{k=1}^N \lambda_k H(x_k^*) = \sum_{k=1}^M \mu_k H(y_k^*)$ of functionals only on elements of the form $y = F(x)$ with $x \in K$.

$$\begin{aligned} \left(\sum_{k=1}^N \lambda_k H(x_k^*) \right) F(x) &= \sum_{k=1}^N \lambda_k F(x_k)^*(F(x)) = \sum_{k=1}^N \lambda_k x_k^*(x) \\ &= \sum_{k=1}^M \mu_k y_k^*(x) = \sum_{k=1}^M \mu_k F(y_k)^*(F(x)) = \left(\sum_{k=1}^M \mu_k H(y_k^*) \right) F(x). \end{aligned}$$

Remark, that according to (5), $\tilde{H}(X^*) = \text{span}H(A) = \text{span}B = Y^*$, so \tilde{H} is surjective, and consequently, by equality of corresponding dimensions, is bijective. Recall, that $\tilde{H}(A) = H(A) = B$, so \tilde{H} maps A to B bijectively. Applying again (5) we deduce that $\tilde{H}(B_{X^*}) = B_{Y^*}$ and X^* is isometric to Y^* . Passing to the duals we deduce that Y^{**} is isometric to X^{**} (with \tilde{H}^* being the corresponding isometry), that is X and Y are isometric. So, B_X and B_Y are two copies of the same compact metric space, and the application of EC-plasticity of compacts [8, Theorem 1.1] completes the proof. \square

Acknowledgement. The author is grateful to her scientific advisor Vladimir Kadets for constant help with this project.

REFERENCES

1. L.E.J. Brouwer, *Beweis der Invarianz des n -dimensionalen Gebiets*, Mathematische Annalen **71** (1912) 305–315.
2. B. Cascales, V. Kadets, J.Orihuela, E.J. Wingler, *Plasticity of the unit ball of a strictly convex Banach space*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **110** (2016),no. 2, 723–727.
3. G. Ding, *On isometric extension problem between two unit spheres*, Sci. China Ser. A **52** (2009) 2069–2083.
4. V. M. Kadets, *A course in functional analysis. Textbook for students of mechanics and mathematics. (Kurs funktsional'nogo analiza. Uchebnoe posobie dlya studentov mekhaniko-matematicheskogo fakulteta) (Russian)*, Khar'kovskij Natsional'nyj Universitet Im. V. N. Karazina, Khar'kov, 2006. <http://page.mi.fu-berlin.de/werner99/kadetsbook/Kadets.Functional.Analysis.pdf>
5. V.Kadets, M.Martín, *Extension of isometries between unit spheres of finite-dimensional polyhedral Banach spaces*, J. Math. Anal. Appl. **386** (2012), 441–447.
6. V.Kadets, O.Zavarzina, *Plasticity of the unit ball of ℓ_1* , Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh. **83** (2017) 4–9.
7. P. Mankiewicz, *On extension of isometries in normed linear spaces*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **20** (1972) 367–371.
8. S.A. Naimpally, Z. Piotrowski, E.J. Wingler, *Plasticity in metric spaces*, J. Math. Anal. Appl. **313** (2006) 38–48.

9. R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1997.
10. D. Tan, X. Huang, R. Liu, *Generalized-lush spaces and the Mazur-Ulam property*, Stud. Math. **219** (2013), no. 2, 139–153.
11. D.Tingley, *Isometries of the unit sphere*, Geom. Dedicata **22** (1987) 371–378.

DEPARTMENT OF MATHEMATICS AND INFORMATICS, V.N. KARAZIN KHARKIV NATIONAL
UNIVERSITY, 61022 KHARKIV, UKRAINE

E-mail address: lesya.nikolchenko@mail.ru